One-Page Review
16.4 (Integrals in other coordinates) March 17, 2016

• Double integral in polar coordinates:
  \[ \int_{D} f(x, y) \, dA = \] 

• Triple integral in cylindrical coordinates:
  \[ \int_{R} f(x, y, z) \, dV = \]

• Triple integral in spherical coordinates:
  \[
  
  
  
  

Problem Set
16.4 (Integrals in other coordinates) March 17, 2016

1. Integrate the function \( f \) over the region \( D \).
   
   (a) \( f(x, y) = x^2 + y^2, \) \( D = \{(x, y) : 1 \leq x^2 + y^2 \leq 4 \} \).
   
   (b) \( f(x, y, z) = z \sqrt{x^2 + y^2}, \) \( D = \{(x, y, z) : x^2 + y^2 \leq z \leq 8 - (x^2 + y^2) \} \).
   
   (c) \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \) \( D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 2z \} \).

2. Show using spherical coordinates that the volume of the sphere of radius \( R \) is what you think it is.

3. Let \( W \) be the region within the cylinder \( x^2 + y^2 = 2 \) between \( z = 0 \) and \( z = \sqrt{x^2 + y^2} \). Calculate the integral of \( f(x, y, z) = x^2 + y^2 \) over \( W \).
Double integral in polar coordinates:
\[ \int \int_D f(x, y) \, dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta \]

Triple integral \( \iiint_R f(x, y, z) \, dV \) in cylindrical coordinates:
\[ \int_{\theta_1}^{\theta_2} \int_{r=r_1(\theta)}^{r=r_2(\theta)} \int_{z=z_1(\theta,r)}^{z=z_2(\theta,r)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta \]

Triple integral in spherical coordinates:
\[ \int_{\theta_1}^{\theta_2} \int_{\phi=\phi_1(\theta)}^{\phi=\phi_2(\theta)} \int_{\rho=\rho_1(\theta,\phi)}^{\rho=\rho_2(\theta,\phi)} f(\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]
1. Integrate the function \( f \) over the region \( D \).

(a) \( f(x, y) = x^2 + y^2 \), \( D = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\} \).

Answer: \( \frac{15\pi}{2} \).

(b) \( f(x, y, z) = z\sqrt{x^2 + y^2} \), \( D = \{(x, y, z) : x^2 + y^2 \leq z \leq 8 - (x^2 + y^2)\} \).

Answer: \( \frac{1024\pi}{15} \).

(c) \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \), \( D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 2z\} \).

Answer: \( \frac{8\pi}{5} \).

2. Show using spherical coordinates that the volume of the sphere of radius \( R \) is what you think it is.

SOLUTION: Integrate the constant function \( f(x, y, z) = 1 \) over the region \( \rho \leq R \) in spherical coordinates:

\[
\int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{4}{3} \pi R^3.
\]

3. Let \( \mathcal{W} \) be the region within the cylinder \( x^2 + y^2 = 2 \) between \( z = 0 \) and \( z = \sqrt{x^2 + y^2} \). Calculate the integral of \( f(x, y, z) = x^2 + y^2 \) over \( \mathcal{W} \).

Answer: \( \frac{8\sqrt{2}\pi}{5} \).